Assignment 8.

This homework is due *Tuesday* Nov 8.

There are total 56 points in this assignment. 47 points is considered 100%. If you go over 47 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 4.1–4.2 in Bartle–Sherbert.

(1) Reminder. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f has limit $L \in \mathbb{R}$ at c if
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.
Below you can find (erroneous!) "definitions" of a limit of a function. In each case describe, exactly which functions "have limit L at c" according to that "definition".
  (a) [4pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
  $\forall \varepsilon > 0\ \forall \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.
  (b) [4pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
  $\exists \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.
  (c) [4pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
  $\exists \delta > 0\ \forall \varepsilon > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.

(2) (Modified 4.1.9) *Use ε–δ definition of limit* to show that
  (a) [2pt] $\lim\limits_{x \to 2} \frac{1}{1-x} = -1$,
  (b) [2pt] $\lim\limits_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$.

(3) (4.1.11) Show that the following limits do not exist:
  (a) [2pt] $\lim\limits_{x \to 0} (x + \operatorname{sgn} x)$,
  (b) [2pt] $\lim\limits_{x \to 0} \sin(1/x^2)$.

(4) (Exercise 4.1.14) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational.
  (a) [3pt] Show that f has limit at $x = 0$ (*Hint*: you can use squeeze theorem).
  (b) [3pt] Prove that if $c \neq 0$, then f does not have limit at c. (*Hint*: you can use sequential criterion.)

— see next page —

(5) [3pt] (Theorem 4.2.4 for difference) *Using ε–δ definition,* prove that limit of functions preserves difference. That is, prove the following:
Let $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A, and f, g be functions on A to $\mathbb{R}$. If $\lim\limits_{x\to c} f = L$, and $\lim\limits_{x\to c} g = M$, then $\lim\limits_{x\to c} f - g = L - M$.

(6) (Exercise 4.2.1,2,3+) Using arithmetic properties of limit, find the following limits.
- (a) [2pt] $\lim\limits_{x\to 1} \frac{x^2+2}{x^2-2}$.
- (b) [2pt] $\lim\limits_{x\to 2} \left(\frac{1}{x+1} - \frac{1}{2x}\right)$.
- (c) [2pt] $\lim\limits_{x\to 0} \frac{(x+1)^2-1}{x}$.
- (d) [2pt] $\lim\limits_{x\to 1} \frac{x^2-1}{x-1}$.
- (e) [2pt] $\lim\limits_{x\to c} \frac{(x-c+1)^2-1}{x-c}$.
- (f) [2pt] $\lim\limits_{x\to 0} \frac{(x^2+1)^2-1}{x}$.
- (g) [3pt] $\lim\limits_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$. (*Hint:* Multiply by conjugate.)

(7)
- (a) [2pt] Sketch graph of $x\sin(1/x)$.
- (b) [3pt] Find $\lim\limits_{x\to 0} x\sin(1/x)$. (*Hint:* Use Squeeze Theorem.)
- (c) [3pt] Sketch graph of $\operatorname{sgn}\sin(1/x)$, establish whether $\lim\limits_{x\to 0} \operatorname{sgn}\sin(1/x)$ exists.

(8) [4pt] (Exercise 4.2.5) Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim\limits_{x\to c} g = 0$. Prove that $\lim\limits_{x\to c} fg = 0$.

Explain why Theorem 4.2.4 (Arithmetic Properties of Limit) cannot be used.